

ON BOST-CONNES TYPE SYSTEMS FOR NUMBER FIELDS

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ABSTRACT. We give a complete description of the phase transition of the Bost-Connes type systems for number fields recently introduced by Connes-Marcolli-Ramachandran and Ha-Paugam. We also introduce a notion of K -lattices and discuss an interpretation of these systems in terms of 1-dimensional K -lattices.

INTRODUCTION

The generalization of the results of Bost and Connes [1] to general number fields has received significant attention for more than ten years, but was only recently formulated in detail as an explicit problem, see [3, Problem 1.1]. We paraphrase here this formulation for easy reference:

Given an algebraic number field K , construct a C^ -dynamical system (\mathcal{A}, σ) such that*

- (i) *the partition function of the system is the Dedekind zeta function of K ;*
- (ii) *the quotient of the idele class group C_K by the connected component D_K of the identity acts as symmetries of the system;*
- (iii) *for each inverse temperature $0 < \beta \leq 1$ there is a unique KMS_β -state;*
- (iv) *for each $\beta > 1$ the action of the symmetry group C_K/D_K on the extremal KMS_β -states is free and transitive;*
- (v) *there is a K -subalgebra \mathcal{A}_0 of \mathcal{A} such that the values of the extremal KMS_∞ -states on elements of \mathcal{A}_0 are algebraic numbers that generate the maximal abelian extension K^{ab} of K ; and*
- (vi) *the Galois action of $\mathcal{G}(K^{ab}/K)$ on these values is realized by the action of C_K/D_K on the extremal KMS_∞ -states via the class field theory isomorphism $s: C_K/D_K \rightarrow \mathcal{G}(K^{ab}/K)$.*

Systems with properties (i)-(iv) have been constructed by several authors for various classes of number fields, see [2, Section 1.4] for a discussion of these constructions and an extensive list of references. However, the last two properties have proven quite elusive. This should not come as a surprise, since a system satisfying (v) and (vi) has the potential to shed light onto Hilbert's 12th problem about the explicit class field theory of K , although this will ultimately depend on the specific expressions obtained for the extremal KMS_∞ -states and the generators of the subalgebra \mathcal{A}_0 . Since imaginary quadratic fields are the only fields beyond \mathbb{Q} for which explicit class field theory is completely understood, it is natural that they should be the first case to be solved, and indeed, Connes, Marcolli, and Ramachandran have produced a complete solution of the problem for these fields, see [3, Theorem 3.1]. It should be noticed also that properties (v) and (vi) are intrinsically related so that the 'right' Galois symmetries and the 'right' arithmetic subalgebra must match each other for the system to have genuine class field theory content. This failed for instance in the system constructed in [9], where it was natural to include certain cyclotomic elements in the arithmetic subalgebra \mathcal{A}_0 , but the Galois action on the corresponding values of extremal KMS_∞ states did not match the symmetry action of the idele class group on these elements, see [9, Theorem 4.4].

In this paper we study a generalization of the system from [3] to all algebraic number fields; this construction is also isomorphic to a particular case, for Shimura data arising from a number field, of a general construction due to Ha and Paugam [5].

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In Section 1 we show how to reduce the study of KMS states of certain restricted groupoid C^* -algebras to measures of the (unrestricted) transformation group (G, X) satisfying a scaling condition. Similar results are well known when G acts freely, see [14]; the key result in this section allows a certain degree of non-freeness and is motivated by our considerations in [11]. We remark that for our applications in Section 2 we could use instead the earlier results from [7] on crossed products by lattice semigroups. However, our present results can be applied to a wider class of systems, e.g. those studied in [9] for fields of class number bigger than one.

Under further assumptions, in Proposition 1.2, we give a natural parametrization of the extremal KMS_β -states in terms of a specific subset of the space X . In Proposition 1.3 we prove a similar result for ground states.

In Section 2 we discuss the dynamical system (\mathcal{A}, σ) . We initially construct the C^* -algebra \mathcal{A} along the lines of [2, 5], using the restricted groupoid obtained from an action of the group of fractional ideals on the cartesian product of $\mathcal{G}(K^{ab}/K)$ by the finite adeles, balanced over the integral ideles. Since we choose to incorporate the class field theory isomorphism in the construction, the symmetry group of the system is $\mathcal{G}(K^{ab}/K)$ itself. We also indicate that \mathcal{A} is a semigroup crossed product of the type discussed in [7], and that, for totally imaginary fields of class number one, the resulting system is isomorphic to the one constructed in [9] using Hecke algebras.

Using the results of Section 1 we show in Theorem 2.1 that for an arbitrary number field K the system (\mathcal{A}, σ) satisfies properties (i) through (iv) above. The description of the symmetry action and classification of the KMS_β -states generalize the corresponding results of [3] and complete the initial results of [5, Section 6]. The argument goes along familiar lines [7, 12], but we include a complete proof to make the paper self-contained.

Finally, we introduce in Section 3 a notion of n -dimensional K -lattices, which generalizes the n -dimensional \mathbb{Q} -lattices from [2] and the 1-dimensional K -lattices for imaginary quadratic fields K from [3]. After discussing some of their basic properties we show in Corollary 3.7 how 1-dimensional K -lattices are related to the systems of Section 2. Therefore these systems can be introduced without using any class field theory data. This may turn out to be useful in verifying properties (v) and (vi), as was the case for \mathbb{Q} and imaginary quadratic fields [2, 3].

1. KMS STATES AND MEASURES

Throughout this section we suppose that G is a countable discrete group acting on a second countable, locally compact, Hausdorff topological space X and that Y is a clopen subset of X satisfying $GY = X$. The C^* -algebra $C_0(X) \rtimes_r G$ is the reduced C^* -algebra of the transformation groupoid $G \times X$. Consider the subgroupoid

$$G \boxtimes Y = \{(g, x) \mid x \in Y, gx \in Y\}$$

and denote by $C_r^*(G \boxtimes Y)$ its reduced C^* -algebra. In other words, $C_r^*(G \boxtimes Y) = \mathbb{1}_Y(C_0(X) \rtimes_r G)\mathbb{1}_Y$, where $\mathbb{1}_Y$ is the characteristic function of Y .

We endow $C_r^*(G \boxtimes Y)$ with the dynamics σ associated to a given homomorphism $N: G \rightarrow (0, +\infty)$, so

$$\sigma_t(f)(g, x) = N(g)^{it} f(g, x)$$

for $t \in \mathbb{R}$ and $f \in C_c(G \boxtimes Y) \subset C_r^*(G \boxtimes Y)$. Recall that a KMS state for σ at inverse temperature $\beta \in \mathbb{R}$, or σ - KMS_β -state, is a σ -invariant state φ such that $\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$ for a and b in a set of σ -analytic elements with dense linear span.

Denote by E the usual conditional expectation from $C_0(X) \rtimes_r G$ onto $C_0(X)$: thus with u_g denoting the element in the multiplier algebra $M(C_0(X) \rtimes_r G)$ corresponding to $g \in G$, we have $E(fu_g) = f$ if $g = e$ and 0 otherwise. Observe that the image under E of the corner $C_r^*(G \boxtimes Y)$ is $C_0(Y)$. By restriction to $C_0(Y)$, a state φ on $C_r^*(G \boxtimes Y)$ gives rise to a Radon probability measure μ , and conversely, a Radon probability measure on Y can be extended via the conditional expectation to a state on $C_r^*(G \boxtimes Y)$. Clearly $(\mu_* \circ E)|_{C_0(Y)} = \mu_*$, but in general it is not true that $(\varphi|_{C_0(Y)}) \circ E$

will be the same as φ . We will show that, under certain combined assumptions on N and the action of G on X , the σ -KMS $_\beta$ -states do arise from their restrictions to $C_0(Y)$, and are thus in one-to-one correspondence with a class of measures on Y characterized by a scaling condition.

Proposition 1.1. *Under the general assumptions on G , X , Y and N listed above, suppose there exist a sequence $\{Y_n\}_{n=1}^\infty$ of Borel subsets of Y and a sequence $\{g_n\}_{n=1}^\infty$ of elements of G such that*

- (i) $\cup_{n=1}^\infty Y_n$ contains the set of points in Y with nontrivial isotropy;
- (ii) $N(g_n) \neq 1$ for all $n \geq 1$;
- (iii) $g_n Y_n = Y_n$ for all $n \geq 1$.

Then for each $\beta \neq 0$ the map $\mu \mapsto \varphi = (\mu_ \circ E)|_{C_r^*(G \boxtimes Y)}$ is an affine isomorphism between Radon measures μ on X satisfying $\mu(Y) = 1$ and the scaling condition*

$$\mu(gZ) = N(g)^{-\beta} \mu(Z) \quad (1.1)$$

for Borel $Z \subset X$, and σ -KMS $_\beta$ -states φ on $C_r^(G \boxtimes Y)$.*

Proof. It is straightforward to check that any measure satisfying the scaling condition extends via E to a KMS $_\beta$ -state.

Conversely, let φ be a KMS $_\beta$ -state. Denote by μ the probability measure on Y defined by $\varphi|_{C_0(Y)}$. Applying the KMS-condition to elements of the form $u_g f u_g^* = f(g^{-1} \cdot)$, it is easy to see that (1.1) is satisfied for Borel $Z \subset Y$ such that $gZ \subset Y$. In particular, $\mu(Y_n) = 0$ by conditions (ii) and (iii).

To show that $\varphi = \mu_* \circ E$ fix $g \neq e$. Let $f \in C_c(Y)$ be such that $g^{-1}(\text{supp } f) \subset Y$. Then $f u_g \in C_r^*(G \boxtimes Y)$ and we have to prove that $\varphi(f u_g) = 0$.

Denote by Y_g the set of points of Y left invariant by g . If $\text{supp } f \cap Y_g = \emptyset$ then we can write f as a finite sum of functions $h_1 h_2$ such that $g(\text{supp } h_1) \cap \text{supp } h_2 = \emptyset$. By the KMS-condition we have

$$\varphi(h_1 h_2 u_g) = \varphi(h_2 u_g h_1) = \varphi(h_2 h_1 (g^{-1} \cdot) u_g) = 0.$$

Therefore $\varphi(f u_g) = 0$.

Assume now that $\text{supp } f \cap Y_g \neq \emptyset$. As $\mu(Y_n) = 0$, by condition (i) we get $\mu(Y_g) = 0$. Hence there exists a norm-bounded sequence $\{f_n\}_n \subset C_c(Y)$ such that $\text{supp } f_n \cap Y_g = \emptyset$, $g^{-1}(\text{supp } f_n) \subset Y$ and $f_n \rightarrow f$ in measure μ . Then $\varphi(f_n u_g) = 0$. On the other hand, by the Cauchy-Schwarz inequality,

$$|\varphi(f u_g) - \varphi(f_n u_g)| \leq \varphi(|f - f_n|)^{1/2} \varphi(u_g^* |f - f_n| u_g)^{1/2} \leq \|f - f_n\|^{1/2} \varphi(|f - f_n|)^{1/2},$$

whence $\varphi(f u_g) = 0$. Therefore $\varphi = \mu_* \circ E$.

To finish the proof it remains to note that the measure μ extends uniquely to a measure on X , which we still denote by μ , such that (1.1) is satisfied for all $g \in G$ and Borel $Z \subset X$. Explicitly, we can write

$$\mu(Z) = \sum_i N(h_i)^\beta \mu(h_i Z \cap Z_i),$$

where $h_i \in G$ and $Z_i \subset Y$ are such that X is the disjoint union of the sets $h_i^{-1} Z_i$, see [10, Lemma 2.2]. \square

Our next goal is to classify measures satisfying the scaling condition. The classification depends on convergence of certain Dirichlet series. More precisely, when S is a subset of G the zeta function associated to S is defined to be

$$\zeta_S(\beta) := \sum_{s \in S} N(s)^{-\beta}.$$

Proposition 1.2. *Assume the hypotheses of Proposition 1.1. Let $\beta \neq 0$, S be a subset of G , and $Y_0 \subset Y$ a nonempty Borel set such that*

- (i) $g Y_0 \cap Y_0 = \emptyset$ for $g \in G \setminus \{e\}$;
- (ii) $S Y_0 \subset Y$;
- (iii) if $g Y_0 \cap Y \neq \emptyset$ then $g \in S$;

(iv) $Y \setminus SU \subset \cup_n Y_n$ for every open set U containing Y_0 ;

(v) $\zeta_S(\beta) < \infty$.

Then

(1) the map $\varphi = \mu_* \circ E \mapsto \zeta_S(\beta)\mu|_{Y_0}$ is an affine isomorphism between the σ -KMS $_\beta$ -states on $C_r^*(G \boxtimes Y)$ and the Borel probability measures on Y_0 ; the inverse map is given by $\nu \mapsto \mu_* \circ E$, where the measure μ on Y is defined by

$$\mu(Z) = \zeta_S(\beta)^{-1} \sum_{s \in S} N(s)^{-\beta} \nu(s^{-1}Z \cap Y_0); \quad (1.2)$$

(2) if μ is the measure on Y defined by a probability measure ν on Y_0 by (1.2), and H_S is the subspace of $L^2(Y, d\mu)$ consisting of functions f such that $f(sy) = f(y)$ for $y \in Y_0$ and $s \in S$, then for $f \in H_S$ we have

$$\|f\|_2^2 = \zeta_S(\beta) \int_{Y_0} |f(y)|^2 d\mu(y); \quad (1.3)$$

furthermore, the orthogonal projection $P: L^2(Y, d\mu) \rightarrow H_S$ is given by

$$Pf|_{sy} = \zeta_S(\beta)^{-1} \sum_{s \in S} N(s)^{-\beta} f(sy) \quad \text{for } y \in Y_0. \quad (1.4)$$

Proof. By Proposition 1.1 any KMS $_\beta$ -state is determined by a Radon measure μ such that $\mu(Y) = 1$ and μ satisfies the scaling condition (1.1). By assumptions (i) and (ii), for such a measure μ we have

$$1 \geq \mu(SY_0) = \sum_{s \in S} N(s)^{-\beta} \mu(Y_0) = \zeta_S(\beta) \mu(Y_0).$$

On the other hand, as $\mu(Y_n) = 0$, by assumption (iv) we have

$$1 = \mu(Y) \leq \mu(SU) \leq \sum_{s \in S} \mu(sU) = \zeta_S(\beta) \mu(U)$$

for any open set U containing Y_0 . By regularity of the measure we conclude that $\zeta_S(\beta) \mu(Y_0) \geq 1$, and hence $\zeta_S(\beta) \mu(Y_0) = 1$. It follows that SY_0 is a subset of Y of full measure. Since μ satisfies the scaling condition, we conclude that μ is completely determined by its restriction to Y_0 .

To finish the proof of (1) we have to construct the inverse map. Let ν be a Borel measure on Y_0 with $\nu(Y_0) = 1$. Similarly to the proof of Proposition 1.1 define a measure μ on X by

$$\mu(Z) = \zeta_S(\beta)^{-1} \sum_{g \in G} N(g)^\beta \nu(gZ \cap Y_0) \quad \text{for Borel } Z \subset X.$$

Then $\zeta_S(\beta)\mu$ extends ν by assumption (i) and satisfies (1.1). Furthermore, by assumptions (ii) and (iii) we have $gY \cap Y_0 \neq \emptyset$ if and only if $g^{-1} \in S$, and in the latter case $Y_0 \subset gY$. It follows that for $Z \subset Y$ we have (1.2). In particular, $\mu(Y) = \zeta_S(\beta)^{-1} \sum_{s \in S} N(s)^{-\beta} \nu(Y_0) = 1$.

Turning to the proof of (2), suppose μ is the measure on Y defined by a probability measure ν on Y_0 by (1.2) and recall that we have already shown that SY_0 is a subset of Y of full μ -measure. Then (2) is a particular case of [10, Lemma 2.9]. For the reader's convenience we sketch a proof.

Equality (1.3) follows from the identity

$$\int_{sY_0} |f|^2 d\mu(y) = N(s)^{-\beta} \int_{Y_0} |f(s \cdot)|^2 d\mu(y),$$

valid for $f \in L^2(Y, d\mu)$, on summing over $s \in S$. Furthermore, as

$$\sum_{s \in S} N(s)^{-\beta} |f(sy)|^2 \geq \zeta_S(\beta) \left| \zeta_S(\beta)^{-1} \sum_{s \in S} N(s)^{-\beta} f(sy) \right|^2$$

the above identity and (1.3) show that the operator T on $L^2(Y, d\mu)$ defined by the right hand side of (1.4) is a contraction. Since $Tf = f$ for $f \in H_S$, and the image of T is H_S , we conclude that $T = P$. \square

In our applications the set S will be a subsemigroup of $\{g \in G \mid N(g) \geq 1\}$ and Y_0 the complement of the union of the sets gY , $g \in S \setminus \{e\}$.

We next give a similar classification of ground states. Recall that a σ -invariant state φ is called a ground state if the holomorphic function $z \mapsto \varphi(a\sigma_z(b))$ is bounded on the upper half-plane for a and b in a set of σ -analytic elements spanning a dense subspace. If a state φ is a weak* limit point of a sequence of states $\{\varphi_n\}_n$ such that φ_n is a σ -KMS $_{\beta_n}$ -state and $\beta_n \rightarrow +\infty$ as $n \rightarrow \infty$, then φ is a ground state. Such ground states are called σ -KMS $_{\infty}$ -states [2].

Proposition 1.3. *Under the general assumptions on G , X , Y and N listed before Proposition 1.1, define $Y_0 = Y \setminus \bigcup_{\{g: N(g) > 1\}} gY$. Assume Y_0 has the property that if $gY_0 \cap Y_0 \neq \emptyset$ for some $g \in G$ then $g = e$. Then the map $\mu \mapsto \mu_* \circ E$ is an affine isomorphism between the Borel probability measures on Y supported on Y_0 and the ground states on $C_r^*(G \boxtimes Y)$.*

Proof. Assume first that μ is a probability measure on Y supported on Y_0 , $\varphi = \mu_* \circ E$. If $a = f_1 u_g$ and $b = f_2 u_h$ with $g^{-1}(\text{supp } f_1), h^{-1}(\text{supp } f_2) \subset Y$, then $E(a\sigma_z(b))$ is nonzero only if $h = g^{-1}$. In the latter case the function $\varphi(a\sigma_z(b)) = N(g)^{-iz} \varphi(ab)$ is clearly bounded on the upper half-plane if $N(g) \leq 1$. So assume $N(g) > 1$. As $u_g f_2 u_g^{-1} = f_2(g^{-1} \cdot)$ is supported on gY , we see that the support of $f_1 f_2(g^{-1} \cdot)$ is contained in $Y \setminus Y_0$, whence $\varphi(a\sigma_z(b)) = 0$.

Conversely, assume φ is a ground state. Let μ be the probability measure on Y defined by $\varphi|_{C_0(Y)}$. Take an element $g \in G$ with $N(g) > 1$. If $f \in C_c(Y \cap g^{-1}Y)$ is positive, $a = u_g f^{1/2}$ and $b = f^{1/2} u_{g^{-1}}$, then the function $z \mapsto \varphi(a\sigma_z(b))$ can be bounded on the upper half-plane only if it is identically zero. Therefore $\varphi(f(g^{-1} \cdot)) = 0$. Hence $\mu(gY \cap Y) = 0$. Thus μ is supported on Y_0 .

It remains to show that $\varphi(fu_g) = 0$ for all $g \neq e$ and $f \in C_c(Y)$ with $g^{-1}(\text{supp } f) \subset Y$. If $x \in \text{supp } f \cap Y_0$ then $g^{-1}x \notin Y_0$ by our assumptions on Y_0 . Hence there exists $h \in G$ with $N(h) > 1$ such that $g^{-1}x \in hY$. This shows that the sets $Y \setminus Y_0$ and ghY with $N(h) > 1$ form an open cover of $\text{supp } f$. Using a partition of unit subordinate to this cover we decompose f into a finite sum of functions with supports contained in these sets. Therefore we may assume that either $\text{supp } f \subset Y \setminus Y_0$ or $g^{-1}(\text{supp } f) \subset hY$ for some h with $N(h) > 1$. In the first case we have $\varphi(fu_g) = 0$ as μ is supported on Y_0 . In the second case write f as a product $f_1 f_2$ of continuous functions with the same support, letting e.g. $f_1 = |f|^{1/2}$ and $f_2 = f|f|^{-1/2}$. Consider the elements $a = f_1 u_{gh}$ and $b = f_2(gh \cdot) u_{h^{-1}}$ of $C_r^*(G \boxtimes Y)$, so that $fu_g = ab$. Since $N(h) > 1$, the function $z \mapsto \varphi(a\sigma_z(b))$ can be bounded on the upper half-plane only if it is identically zero. Therefore $\varphi(fu_g) = 0$. \square

2. BOST-CONNES SYSTEMS FOR NUMBER FIELDS

Suppose K is an algebraic number field with subring of integers \mathcal{O} . Recall some notation. Denote by V_K the set of places of K , and by $V_{K,f} \subset V_K$ the subset of finite places. For $v \in V_K$ denote by K_v the corresponding completion of K . If v is finite, let \mathcal{O}_v be the closure of \mathcal{O} in K_v . The ring of finite integral adeles is $\hat{\mathcal{O}} = \prod_{v \in V_{K,f}} \mathcal{O}_v$, and $\mathbb{A}_{K,f} = K \otimes_{\mathcal{O}} \hat{\mathcal{O}}$ is the ring of finite adeles. Denoting by $K_{\infty} = \prod_{v \in V_K} K_v$ the completion of K at all infinite places, we get the ring $\mathbb{A}_K = K_{\infty} \times \mathbb{A}_{K,f}$ of adeles. The idele group is $I_K = \mathbb{A}_K^*$.

Consider the topological space $\mathcal{G}(K^{ab}/K) \times \mathbb{A}_{K,f}$, where $\mathcal{G}(K^{ab}/K)$ is the Galois group of the maximal abelian extension of K . On this space there is an action of the group $\mathbb{A}_{K,f}^*$ of finite ideles, via the Artin map $s: I_K \rightarrow \mathcal{G}(K^{ab}/K)$ on the first component and via multiplication on the second component:

$$j(\gamma, m) = (\gamma s(j)^{-1}, jm) \quad \text{for } j \in \mathbb{A}_{K,f}^*, \quad \gamma \in \mathcal{G}(K^{ab}/K), \quad m \in \mathbb{A}_{K,f}.$$

Following [3] we consider the quotient space

$$X := \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f}$$

in which the direct product is balanced over the compact open subgroup of integral ideles $\hat{\mathcal{O}}^* \subset \mathbb{A}_{K,f}^*$, in the sense that one takes the quotient by the action given by $u(\gamma, m) = (\gamma s(u)^{-1}, um)$ for $u \in \hat{\mathcal{O}}^*$. This enables a quotient action of the quotient group $\mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*$, which is isomorphic to the (discrete) group J_K of fractional ideals in K . We remark that the space X is isomorphic to the one that arises from the construction of Ha and Paugam when applied to the Shimura data associated to a number field, see [5, Definition 5.5].

Finally we restrict to the clopen subset $Y := \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}$ of X , and we consider the dynamical system $(C_r^*(J_K \boxtimes Y), \sigma)$, in which the dynamics σ is defined in terms of the absolute norm $N: J_K \rightarrow (0, +\infty)$. Denote by $J_K^+ \subset J_K$ the subsemigroup of integral ideals, and recall that the norm of such an ideal \mathfrak{a} is given by $|\mathcal{O}/\mathfrak{a}|$. Remark that by Theorem 2.1 and Theorem 2.4 of [8] the corner $C_r^*(J_K \boxtimes Y) = \mathbb{1}_Y(C_0(X) \rtimes J_K)\mathbb{1}_Y$ is the semigroup crossed product $C(Y) \rtimes J_K^+$.

In this situation the zeta function of the semigroup J_K^+ is precisely the Dedekind zeta function $\zeta_K(\beta) = \sum_{\mathfrak{a} \in J_K^+} N(\mathfrak{a})^{-\beta}$; it converges for $\beta > 1$ and diverges for $\beta \in (0, 1]$.

Theorem 2.1. *For the system $(C(\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}) \rtimes J_K^+, \sigma)$ we have:*

- (i) *for $\beta < 0$ there are no KMS_β -states;*
- (ii) *for each $0 < \beta \leq 1$ there is a unique KMS_β -state;*
- (iii) *for each $1 < \beta < \infty$ the extremal KMS_β -states are indexed by $Y_0 := \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}^* \cong \mathcal{G}(K^{ab}/K)$, with the state corresponding to $w \in Y_0$ given by*

$$\varphi_{\beta,w}(f) = \frac{1}{\zeta_K(\beta)} \sum_{\mathfrak{a} \in J_K^+} N(\mathfrak{a})^{-\beta} f(\mathfrak{a}w) \quad \text{for } f \in C(\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}); \quad (2.1)$$

- (iv) *the ground states of the system are KMS_∞ -states, and the extremal ground states are indexed by Y_0 , with the state corresponding to $w \in Y_0$ given by $\varphi_{\infty,w}(f) = f(w)$.*

Proof. We apply Proposition 1.1 to $G = J_K$, $X = \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f}$ and $Y = \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}$. If the image of a point $(\alpha, a) \in \mathcal{G}(K^{ab}/K) \times \mathbb{A}_{K,f}$ in X has nontrivial isotropy then $a_v = 0$ for some v , since this is true already for the action of $J_K = \mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*$ on $\mathbb{A}_{K,f}/\hat{\mathcal{O}}^*$. Therefore for the sequence $\{(g_n, Y_n)\}_n$ we can take the pairs (\mathfrak{p}_v, Y_v) indexed by the finite places v , where \mathfrak{p}_v is the prime ideal of \mathcal{O} corresponding to v and $Y_v \subset Y$ consists of the images in X of all pairs $(\alpha, a) \in \mathcal{G}(K^{ab}/K) \times \hat{\mathcal{O}}$ with $a_v = 0$. By Proposition 1.1 we conclude that the KMS_β -states for $\beta \neq 0$ correspond to the measures μ on X such that $\mu(Y) = 1$ and μ satisfies the scaling condition (1.1).

Clearly there are no such measures for $\beta < 0$, since otherwise the inclusion $\mathfrak{a}Y \subset Y$ would imply $N(\mathfrak{a})^{-\beta} \leq 1$. This proves (i).

To prove part (iii) notice that $S = J_K^+$ and $Y_0 = \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}^* \subset Y$ satisfy conditions (i), (ii) and (iii) of Proposition 1.2. In order to verify condition (iv) let $A \subset V_{K,f}$ be a finite set and denote by \mathcal{O}_A the product of \mathcal{O}_v over $v \in A$, and by $\hat{\mathcal{O}}_A$ the product of \mathcal{O}_v over $v \notin A$, so that $\hat{\mathcal{O}} = \mathcal{O}_A \times \hat{\mathcal{O}}_A$. Consider the open subset

$$W_A = \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} (\mathcal{O}_A^* \times \hat{\mathcal{O}}_A)$$

of Y . The intersection of these sets over all finite A coincides with Y_0 . Since Y is compact and the sets W_A are closed, it follows that any neighborhood of Y_0 contains W_A for some A . The complement of $J_K^+ W_A$ in Y consists of the images of points $(\alpha, a) \in \mathcal{G}(K^{ab}/K) \times \hat{\mathcal{O}}$ such that $a_v = 0$ for some $v \in A$, so it is covered by the sets Y_v , $v \in A$, introduced above. Thus by Proposition 1.2 for each $\beta > 1$ there is a one-to-one affine correspondence between the KMS_β -states and the probability

measures on Y_0 . In particular, the extremal KMS_β -states correspond to points of Y_0 via (2.1), which is a particular case of (1.2). This finishes the proof of part (iii).

Part (iv) follows from (iii) and Proposition 1.3.

Turning to (ii), we shall first explicitly construct for each $\beta \in (0, 1]$ a measure μ_β on X such that $\mu_\beta(Y) = 1$ and μ_β satisfies the scaling condition (1.1). Define μ_β as the push-forward of the product measure $\mu_{\mathcal{G}} \times \prod_{v \in V_{K,f}} \mu_{\beta,v}$ on $\mathcal{G}(K^{ab}/K) \times \mathbb{A}_{K,f}$, where $\mu_{\mathcal{G}}$ is the normalized Haar measure on $\mathcal{G}(K^{ab}/K)$ and the measures $\mu_{\beta,v}$ on K_v are defined as follows. The measure $\mu_{1,v}$ is the additive Haar measure on K_v normalized by $\mu_{1,v}(\mathcal{O}_v) = 1$. The measure $\mu_{\beta,v}$ is defined so that it is equivalent to $\mu_{1,v}$ and

$$\frac{d\mu_{\beta,v}}{d\mu_{1,v}}(a) = \frac{1 - N(\mathfrak{p}_v)^{-\beta}}{1 - N(\mathfrak{p}_v)^{-1}} \|a\|_v^{\beta-1},$$

where $\|\cdot\|_v$ is the normalized valuation in the class v , so $\|\pi\|_v = N(\mathfrak{p}_v)^{-1}$ for any uniformizing parameter $\pi \in \mathfrak{p}_v$. Equivalently, $\mu_{\beta,v}$ is the unique measure on K_v such that the restriction of $\mu_{\beta,v}$ to \mathcal{O}_v^* is the (multiplicative) Haar measure normalized by $\mu_{\beta,v}(\mathcal{O}_v^*) = 1 - N(\mathfrak{p}_v)^{-\beta}$, and $\mu_{\beta,v}(\pi Z) = N(\mathfrak{p}_v)^{-\beta} \mu_{\beta,v}(Z)$.

To show that the measure μ_β is unique it suffices to show that the action of J_K on (X, μ) is ergodic for every measure μ on X such that $\mu(Y) = 1$ and μ satisfies the scaling condition (1.1). Indeed, the set of such measures is affine, so if all measures are ergodic the set must consist of one point.

Equivalently, we have to show that the subspace H of $L^2(Y, d\mu)$ of J_K^+ -invariant functions consists of scalars. Denote by P the projection onto this space. It is enough to compute how P acts on the pull-backs of functions on $\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathcal{O}_A$ for finite $A \subset V_{K,f}$. Denote by $J_{K,A}^+$ the unital subsemigroup of J_K^+ generated by \mathfrak{p}_v , $v \in A$. Modulo a set of measure zero $\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathcal{O}_A$ is the union of the sets $\mathfrak{a}(\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathcal{O}_A^*)$, $\mathfrak{a} \in J_{K,A}^+$. The compact set $\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathcal{O}_A^*$ is a group isomorphic to $\mathcal{G}(K^{ab}/K)/s(\hat{\mathcal{O}}_A^*)$. Therefore it suffices to compute Pf for the pull-back f of the function

$$\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathcal{O}_A \ni a \mapsto \begin{cases} \tilde{\chi}(\mathfrak{a}^{-1}a), & \text{if } a \in \mathfrak{a}(\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathcal{O}_A^*), \\ 0, & \text{otherwise,} \end{cases}$$

where $\tilde{\chi}$ is a character of $\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathcal{O}_A^*$. The character $\tilde{\chi}$ is defined by a Dirichlet character $\chi \bmod \mathfrak{m}$ with $\mathfrak{m} \in J_{K,A}^+$.

For a finite set $B \subset V_{K,f}$ denote by P_B the projection onto the subspace $H_B \subset L^2(Y, d\mu)$ of $J_{K,B}^+$ -invariant functions. Apply Proposition 1.2(2) with $G = J_{K,B} := (J_{K,B}^+)^{-1} J_{K,B}^+$, $S = J_{K,B}^+$ and $Y_0 = W_B = \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} (\mathcal{O}_B^* \times \hat{\mathcal{O}}_B)$. Note that $\zeta_{J_{K,B}^+}(\beta) = \prod_{v \in B} (1 - N(\mathfrak{p}_v)^{-\beta})^{-1}$. Furthermore, for $\mathfrak{b} \in J_{K,B}^+$ the set $\mathfrak{b}W_B$ intersects the support of f only if $\mathfrak{a}|\mathfrak{b}$ and the ideals \mathfrak{a} and $\mathfrak{b}\mathfrak{a}^{-1}$ are relatively prime, or equivalently, $\mathfrak{a} \in J_{K,B}^+$ and $\mathfrak{b} \in \mathfrak{a}J_{K,B \setminus A}^+$. Therefore, assuming $A \subset B$, by (1.4) we get

$$\begin{aligned} P_B f|_{J_{K,B}^+} &= \prod_{v \in B} (1 - N(\mathfrak{p}_v)^{-\beta}) \sum_{\mathfrak{c} \in J_{K,B \setminus A}^+} N(\mathfrak{a}\mathfrak{c})^{-\beta} \tilde{\chi}(\mathfrak{c}\mathfrak{a}) \\ &= N(\mathfrak{a})^{-\beta} \tilde{\chi}(\mathfrak{a}) \prod_{v \in B} (1 - N(\mathfrak{p}_v)^{-\beta}) \sum_{\mathfrak{c} \in J_{K,B \setminus A}^+} N(\mathfrak{c})^{-\beta} \chi(\mathfrak{c}) \\ &= N(\mathfrak{a})^{-\beta} \tilde{\chi}(\mathfrak{a}) \frac{\prod_{v \in B} (1 - N(\mathfrak{p}_v)^{-\beta})}{\prod_{v \in B \setminus A} (1 - \chi(\mathfrak{p}_v) N(\mathfrak{p}_v)^{-\beta})} \end{aligned}$$

for $a \in W_B$. If χ is trivial we see that $P_B f$ is constant, and hence so is Pf . On the other hand, for nontrivial χ we get

$$\|Pf\|_2 = \lim_B \|P_B f\|_2 = N(\mathfrak{a})^{-\beta} \lim_B \frac{\prod_{v \in B} |1 - N(\mathfrak{p}_v)^{-\beta}|}{\prod_{v \in B \setminus A} |1 - \chi(\mathfrak{p}_v) N(\mathfrak{p}_v)^{-\beta}|}.$$

The right hand side divided by $N(\mathfrak{a})^{-\beta}$ is an increasing function in β on $(0, +\infty)$. For $\beta > 1$ it equals $|L(\chi, \beta)|/\zeta_K(\beta)$. As $L(\chi, \cdot)$ does not have a pole at 1, see e.g. [13, Lemma 13.3], we conclude that the right hand side is zero for $\beta \in (0, 1]$. Therefore in either case we see that Pf is constant. \square

Remark 2.2.

(i) There is an obvious action of the Galois group $\mathcal{G}(K^{ab}/K)$ of the maximal abelian extension of K on Y , given by $\alpha(\gamma, m) = (\alpha\gamma, m)$, and this gives rise to an action of $\mathcal{G}(K^{ab}/K)$ as symmetries of $(C_r^*(J_K \boxtimes Y), \sigma)$. This action is clearly free and transitive on the set Y_0 parametrizing the extreme KMS_β -states.

(ii) It is known [5] and easy to check that the partition function of our system is the Dedekind zeta function. More precisely, if $\varphi_{\beta, w}$ is an extremal KMS_β -state for some $\beta > 1$, and $H_{\beta, w}$ is the generator of the canonical one-parameter unitary group implementing σ in the GNS-representation of $\varphi_{\beta, w}$, then $\text{Tr}(e^{-\beta H_{\beta, w}}) = \zeta_K(\beta)$.

(iii) For totally imaginary fields of class number one the C^* -algebra $C_r^*(J_K \boxtimes Y)$ described above is isomorphic to the Hecke C^* -algebra $C^*(\Gamma_K; \Gamma_{\mathcal{O}})$ studied in [9]. To see this, observe first that $\mathcal{G}(K^{ab}/K) \cong \mathbb{A}_{K, f}^*/\overline{K^*} \cong \hat{\mathcal{O}}^*/\overline{\mathcal{O}^*}$. It follows that $Y = \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}$ can be identified with $\hat{\mathcal{O}}/\overline{\mathcal{O}^*}$. From [9, Definition 2.2] and the ensuing discussion, multiplication by an extreme inverse different transforms this identification into a homeomorphism of the orbit space $\Omega = \mathcal{D}^{-1}/\overline{\mathcal{O}^*}$ and Y . It is then easy to check that the multiplicative action of $J_K^+ \cong \mathcal{O}^\times/\mathcal{O}^*$ on $C(\Omega)$ described in [9, Proposition 2.4] corresponds to the action of $J_K^+ \cong \mathbb{A}_{K, f}^*/\hat{\mathcal{O}}^*$ inherited by $C(Y)$ from the original transformation group. By [9, Theorem 2.5] it follows that the Hecke C^* -algebra $C^*(\Gamma_K; \Gamma_{\mathcal{O}})$ is isomorphic to $C(Y) \rtimes J_K^+ \cong C_r^*(J_K \boxtimes Y)$. The isomorphism respects the semigroup of isometries and thus the dynamics arising from the norm, but the Galois group action is changed via the balancing over $\hat{\mathcal{O}}^*$, and this resolves the incompatibility pointed out in [9, Theorem 4.4].

For higher class numbers the Hecke C^* -algebra constructed in [9] is a semigroup crossed product by the semigroup of principal ideals so it is essentially different from the one studied here.

3. K -LATTICES

In this section we define n -dimensional K -lattices and interpret the BC-systems for number fields in terms of K -lattices.

Recall the following definition given by Connes and Marcolli [2]. An n -dimensional \mathbb{Q} -lattice is a pair (L, φ) , where $L \subset \mathbb{R}^n$ is a lattice and $\varphi: \mathbb{Q}^n/\mathbb{Z}^n \rightarrow \mathbb{Q}L/L$ is a homomorphism. The notion of a 1-dimensional K -lattice for an imaginary quadratic field K is analyzed in [3]. In what follows we generalize K -lattices to arbitrary number fields and dimensions. We refer to [4] for a related discussion of the function fields case, see also [6].

Recall that we denote by K_∞ the completion of K at all infinite places, so $K_\infty \cong \mathbb{R}^{[K: \mathbb{Q}]}$ as a topological group under addition. By an n -dimensional \mathcal{O} -lattice we mean a lattice L in K_∞^n such that $\mathcal{O}L = L$.

Definition 3.1. An n -dimensional K -lattice is a pair (L, φ) , where $L \subset K_\infty^n$ is an n -dimensional \mathcal{O} -lattice and $\varphi: K^n/\mathcal{O}^n \rightarrow KL/L$ is an \mathcal{O} -module map.

The simplest example of an n -dimensional \mathcal{O} -lattice is \mathcal{O}^n . Since $K^n = \mathbb{Q}\mathcal{O}^n$, any two finitely generated \mathcal{O} -submodules of K^n of rank n are commensurable, in particular, any such module is an \mathcal{O} -lattice. Furthermore, a submodule of K^n of rank $m < n$ is an abelian group of rank $m[K: \mathbb{Q}]$, so

it cannot be a lattice in K_∞^n . Thus for submodules of K^n we get the usual definition of an \mathcal{O} -lattice: an \mathcal{O} -submodule $M \subset K^n$ is an n -dimensional \mathcal{O} -lattice if and only if it is finitely generated and has rank n .

We now want to give a parametrization of the set of n -dimensional \mathcal{O} -lattices. For this recall that there exists a one-to-one correspondence between finitely generated \mathcal{O} -submodules of K^n of rank n and $\hat{\mathcal{O}}$ -submodules $\mathcal{L} = \prod_{v \in V_{K,f}} L_v \subset \mathbb{A}_{K,f}^n$ such that L_v is a compact open \mathcal{O}_v -submodule of K_v^n with $L_v = \mathcal{O}_v^n$ for all but a finite number of places v . Namely, starting from an \mathcal{O} -lattice define \mathcal{L} as its closure. The inverse map is $\mathcal{L} \mapsto \cap_v (L_v \cap K^n)$. Hence, given an element $s = (s_\infty, s_f) \in \mathrm{GL}_n(\mathbb{A}_K) = \mathrm{GL}_n(K_\infty) \times \mathrm{GL}_n(\mathbb{A}_{K,f})$, we get an \mathcal{O} -lattice $s_f \hat{\mathcal{O}}^n \cap K^n$ in K^n , and then an \mathcal{O} -lattice $s_\infty^{-1}(s_f \hat{\mathcal{O}}^n \cap K^n)$ in K_∞^n .

Lemma 3.2. *The map $\mathrm{GL}_n(\mathbb{A}_K) \ni s \mapsto s_\infty^{-1}(s_f \hat{\mathcal{O}}^n \cap K^n)$ induces a bijection between*

$$\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K) / \mathrm{GL}_n(\hat{\mathcal{O}})$$

and the set of n -dimensional \mathcal{O} -lattices.

Proof. It is easy to see that the map from $\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K) / \mathrm{GL}_n(\hat{\mathcal{O}})$ to \mathcal{O} -lattices is well-defined. To see that it is injective, assume $r_\infty^{-1}(r_f \hat{\mathcal{O}}^n \cap K^n) = s_\infty^{-1}(s_f \hat{\mathcal{O}}^n \cap K^n)$ for some $r, s \in \mathrm{GL}_n(\mathbb{A}_K)$. Multiplying by K we get $r_\infty^{-1}K^n = s_\infty^{-1}K^n$, so $g := s_\infty r_\infty^{-1} \in \mathrm{GL}_n(K)$. Taking the closure we get from $g(r_f \hat{\mathcal{O}}^n \cap K^n) = s_f \hat{\mathcal{O}}^n \cap K^n$ that $gr_f \hat{\mathcal{O}}^n = s_f \hat{\mathcal{O}}^n$. Hence $gr_f u = s_f$ for some $u \in \mathrm{GL}_n(\hat{\mathcal{O}})$. Since also $gr_\infty = s_\infty$, this means that s is in a $\mathrm{GL}_n(K)$ - $\mathrm{GL}_n(\hat{\mathcal{O}})$ -orbit of r , so the map is injective.

To prove surjectivity, take an \mathcal{O} -lattice $L \subset K_\infty^n$. We have $KL = \mathbb{Q}L \cong \mathbb{Q}^{[K:\mathbb{Q}]}$, so $\dim_K KL = n$. In particular, L is a finitely generated \mathcal{O} -module of rank n . Therefore it suffices to show that there exists $g \in \mathrm{GL}_n(K_\infty)$ such that $gL \subset K^n$. Let e_1, \dots, e_n be a basis of KL over K . Since $KL = \mathbb{Q}L$ is dense in K_∞^n , the image of KL under the projection $K_\infty^n \rightarrow K_v^n$ is dense in K_v^n for any infinite place v . It follows that the images of e_1, \dots, e_n are linearly independent over K_v . So there exists $g_v \in \mathrm{GL}_n(K_v)$ which maps these images onto the standard basis of K_v^n . Then $g = (g_v)_{v|\infty}$ is an element in $\mathrm{GL}_n(K_\infty)$ mapping e_1, \dots, e_n onto the standard basis of K_∞^n , so that $gKL = K^n$. \square

For $s \in \mathrm{GL}_n(\mathbb{A}_K)$ and $t \in \mathrm{Mat}_n(\hat{\mathcal{O}})$ consider the \mathcal{O} -lattice $L = s_f \hat{\mathcal{O}}^n \cap K^n$. The map $s_f t: \mathbb{A}_{K,f}^n \rightarrow \mathbb{A}_{K,f}^n$ maps $\hat{\mathcal{O}}^n$ into $s_f \hat{\mathcal{O}}^n$, hence induces an $\hat{\mathcal{O}}$ -module map $\mathbb{A}_{K,f}^n / \hat{\mathcal{O}}^n \rightarrow \mathbb{A}_{K,f}^n / s_f \hat{\mathcal{O}}^n$. Then there exists a unique \mathcal{O} -module map $\varphi: K^n / \mathcal{O}^n \rightarrow KL / L$ such that the diagram

$$\begin{array}{ccc} \mathbb{A}_{K,f}^n / \hat{\mathcal{O}}^n & \xrightarrow{s_f t} & \mathbb{A}_{K,f}^n / s_f \hat{\mathcal{O}}^n \\ \uparrow & & \uparrow \\ K^n / \mathcal{O}^n & \xrightarrow{\varphi} & KL / L \end{array}$$

commutes, where the vertical arrows are the canonical isomorphisms defined by the inclusions $K^n \subset \mathbb{A}_{K,f}^n$, $KL \subset \mathbb{A}_{K,f}^n$. We shall also denote φ by $[s_f t]$. Thus (L, φ) is a K -lattice. Therefore $(s_\infty^{-1}L, s_\infty^{-1}\varphi)$ is also a K -lattice, which we denote by $[(s, t)]$.

Lemma 3.3. *The map $\mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\hat{\mathcal{O}}) \ni (s, t) \rightarrow [(s, t)] = (s_\infty^{-1}(s_f \hat{\mathcal{O}}^n \cap K^n), s_\infty^{-1}[s_f t])$ induces a bijection between*

$$\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K) \times_{\mathrm{GL}_n(\hat{\mathcal{O}})} \mathrm{Mat}_n(\hat{\mathcal{O}})$$

and the set of n -dimensional K -lattices.

Proof. By Lemma 3.2 we only need to check that any \mathcal{O} -module map $\mathbb{A}_{K,f}^n / \hat{\mathcal{O}}^n \rightarrow \mathbb{A}_{K,f}^n / s_f \hat{\mathcal{O}}^n$, where $s_f \in \mathrm{GL}_n(\mathbb{A}_{K,f})$, is defined by the matrix $s_f t$ for a unique $t \in \mathrm{Mat}_n(\hat{\mathcal{O}})$. It suffices to consider $s_f = 1$. The problem then reduces to showing that any \mathcal{O} -module map $K_v / \mathcal{O}_v \rightarrow K_v / \mathcal{O}_v$ is given by multiplication by a unique element of \mathcal{O}_v . If π is a uniformizing parameter in \mathcal{O}_v , then any \mathcal{O} -module

map $\mathcal{O}_v \pi^{-m} / \mathcal{O}_v \rightarrow K_v / \mathcal{O}_v$ is determined by the image of π^{-m} , so it is given by multiplication by an element in \mathcal{O}_v which is uniquely determined modulo $\mathcal{O}_v \pi^m$. Since \mathcal{O}_v is complete in the (π) -adic topology, this gives the result. \square

Notice that we have shown in particular that for any K -lattice (L, φ) with $L \subset K^n$ the homomorphism φ lifts to a unique $\mathbb{A}_{K,f}$ -module map $\tilde{\varphi}: \mathbb{A}_{K,f}^n \rightarrow \mathbb{A}_{K,f}^n$.

Definition 3.4. Two n -dimensional K -lattices (L_1, φ_1) and (L_2, φ_2) are called commensurable if the lattices L_1 and L_2 are commensurable and $\varphi_1 = \varphi_2$ modulo $L_1 + L_2$.

If L_1 and L_2 are commensurable then $KL_1 = \mathbb{Q}L_1 = \mathbb{Q}L_2 = KL_2$. In particular, if $L_1 \subset K^n$ then also $L_2 \subset K^n$. It is clear that then the lifting of the composition of the homomorphisms $\varphi_1: K^n / \mathcal{O}^n \rightarrow KL_1 / L_1$ and $KL_1 / L_1 \rightarrow K(L_1 + L_2) / (L_1 + L_2)$ coincides with $\tilde{\varphi}_1$. Therefore two K -lattices (L_1, φ_1) and (L_2, φ_2) with $L_1, L_2 \subset K^n$ are commensurable if and only if $\tilde{\varphi}_1 = \tilde{\varphi}_2$. This implies that commensurability is an equivalence relation.

Denote the equivalence relation of commensurability of n -dimensional K -lattices by $\mathcal{R}_{K,n}$. Consider now the action of $\mathrm{GL}_n(\mathbb{A}_{K,f})$ on $\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\mathbb{A}_{K,f})$ defined by

$$g(s, t) = (sg^{-1}, gt).$$

Define a subgroupoid

$$\mathrm{GL}_n(\mathbb{A}_{K,f}) \boxtimes (\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\hat{\mathcal{O}})) = \{(g, s, t) \mid t \in \mathrm{Mat}_n(\hat{\mathcal{O}}), gt \in \mathrm{Mat}_n(\hat{\mathcal{O}})\}$$

of the transformation groupoid $\mathrm{GL}_n(\mathbb{A}_{K,f}) \times (\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\mathbb{A}_{K,f}))$. We have a groupoid homomorphism

$$\mathrm{GL}_n(\mathbb{A}_{K,f}) \boxtimes (\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\hat{\mathcal{O}})) \rightarrow \mathcal{R}_{K,n}$$

defined by

$$(g, s, t) \mapsto [(sg^{-1}, gt)], [(s, t)]. \quad (3.1)$$

To see that $[(s, t)]$ and $[(sg^{-1}, gt)]$ are indeed commensurable recall that by definition we have $[(s, t)] = (s_\infty^{-1}(s_f \hat{\mathcal{O}}^n \cap K^n), s_\infty^{-1}[s_f t])$ and $[(sg^{-1}, gt)] = (s_\infty^{-1}(s_f g^{-1} \hat{\mathcal{O}}^n \cap K^n), s_\infty^{-1}[s_f t])$.

By Lemma 3.3 to make the above homomorphism injective we have to factor out the action of $\mathrm{GL}_n(\hat{\mathcal{O}})$. Consider the action of $\mathrm{GL}_n(\hat{\mathcal{O}}) \times \mathrm{GL}_n(\hat{\mathcal{O}})$ on $\mathrm{GL}_n(\mathbb{A}_{K,f}) \boxtimes (\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\hat{\mathcal{O}}))$ defined by

$$(u_1, u_2)(g, s, t) = (u_1 g u_2^{-1}, s u_2^{-1}, u_2 t),$$

and denote by

$$\mathrm{GL}_n(\hat{\mathcal{O}}) \backslash \mathrm{GL}_n(\mathbb{A}_{K,f}) \boxtimes_{\mathrm{GL}_n(\hat{\mathcal{O}})} (\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\hat{\mathcal{O}}))$$

the quotient space.

Proposition 3.5. *The map (3.1) induces a bijection between*

$$\mathrm{GL}_n(\hat{\mathcal{O}}) \backslash \mathrm{GL}_n(\mathbb{A}_{K,f}) \boxtimes_{\mathrm{GL}_n(\hat{\mathcal{O}})} (\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\hat{\mathcal{O}}))$$

and $\mathcal{R}_{K,n}$.

Proof. By Lemma 3.3 the map

$$\mathrm{GL}_n(\hat{\mathcal{O}}) \backslash \mathrm{GL}_n(\mathbb{A}_{K,f}) \boxtimes_{\mathrm{GL}_n(\hat{\mathcal{O}})} (\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\hat{\mathcal{O}})) \rightarrow \mathcal{R}_{K,n}$$

is well-defined and injective. To prove surjectivity we have to show that if $(L, \varphi) = [(s, t)]$ is a K -lattice then any commensurable K -lattice is of the form $[(sg^{-1}, gt)]$ for some $g \in \mathrm{GL}_n(\mathbb{A}_{K,f})$. We may assume that $L \subset K^n$ and then that $s_\infty = 1$. Then by Lemma 3.3 and the discussion following Definition 3.4 any commensurable K -lattice is of the form $[(q, r)]$ with $q_\infty = 1$ and $q_f r = s_f t$. Letting $g = q_f^{-1} s_f$ we get $(q, r) = (sg^{-1}, gt)$. \square

Remark 3.6. In the case $K = \mathbb{Q}$, or more generally for fields with class number one, there is a better description due to the fact that any \mathbb{Z} -lattice is free. Indeed, by freeness we have $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q},f}) = \mathrm{GL}_n^+(\mathbb{Q})\mathrm{GL}_n(\hat{\mathbb{Z}})$, where $\mathrm{GL}_n^+(\mathbb{Q})$ is the group of rational matrices with positive determinant. It follows that any $\mathrm{GL}_n(\hat{\mathbb{Z}}) \times \mathrm{GL}_n(\hat{\mathbb{Z}})$ -orbit in $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q},f}) \times (\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}) \times \mathrm{Mat}_n(\hat{\mathbb{Z}}))$ has a representative in $\mathrm{GL}_n^+(\mathbb{Q}) \times ((\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n^+(\mathbb{Q})) \times \mathrm{Mat}_n(\hat{\mathbb{Z}}))$. Furthermore, the map

$$\mathrm{GL}_n^+(\mathbb{R}) \times \mathrm{GL}_n^+(\mathbb{Q}) \rightarrow \mathrm{GL}_n^+(\mathbb{R}), \quad (g, h) \mapsto h^{-1}g,$$

induces a bijection between $\mathrm{GL}_n^+(\mathbb{Q}) \backslash (\mathrm{GL}_n^+(\mathbb{R}) \times \mathrm{GL}_n^+(\mathbb{Q}))$ onto $\mathrm{GL}_n^+(\mathbb{R})$. One may then conclude that $\mathcal{R}_{\mathbb{Q},n}$ can be identified with

$$\mathrm{SL}_n(\mathbb{Z}) \backslash \mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_{\mathrm{SL}_n(\mathbb{Z})} (\mathrm{GL}_n^+(\mathbb{R}) \times \mathrm{Mat}_n(\hat{\mathbb{Z}})),$$

where the action of $\mathrm{SL}_n(\mathbb{Z}) \times \mathrm{SL}_n(\mathbb{Z})$ on $\mathrm{GL}_n^+(\mathbb{Q}) \times \mathrm{GL}_n^+(\mathbb{R}) \times \mathrm{Mat}_n(\hat{\mathbb{Z}})$ is given by

$$(\gamma_1, \gamma_2)(g, h, m) = (\gamma_1 g \gamma_2^{-1}, \gamma_2 h, \gamma_2 m).$$

Consider now the case $n = 1$ (and K arbitrary). Then we conclude that there is a bijection between $\mathcal{R}_{K,1}$ and the subgroupoid

$$(\mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*) \boxtimes ((\mathbb{A}_K^*/K^*) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}})$$

of the transformation groupoid $(\mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*) \times ((\mathbb{A}_K^*/K^*) \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f})$. We have an action, called the scaling action, of K_∞^* on K -lattices: if (L, φ) is a K -lattice and $k \in K_\infty^*$ then $k(L, \varphi) = (kL, k\varphi)$. It defines an action of K_∞^* on $\mathcal{R}_{K,1}$. In our transformation groupoid picture of $\mathcal{R}_{K,1}$ it corresponds to the action of K_∞^* by multiplication on \mathbb{A}_K^*/K^* . Denote by $(K_\infty^*)^\circ$ the connected component of the identity in K_∞^* . Then we get the following result.

Corollary 3.7. *The quotient of the equivalence relation $\mathcal{R}_{K,1}$ of commensurability of 1-dimensional K -lattices by the scaling action of the connected component of the identity in K_∞^* is a groupoid that is isomorphic to*

$$(\mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*) \boxtimes ((\mathbb{A}_K^*/K^*(K_\infty^*)^\circ) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}).$$

Recalling that $\mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^* \cong J_K$ and $\mathbb{A}_K^*/\overline{K^*(K_\infty^*)^\circ} \cong \mathcal{G}(K^{ab}/K)$ by class field theory, we see that the above groupoid is almost the same that we used to define the BC-system. The small nuance is that when we put $\mathcal{G}(K^{ab}/K)$ in our topological groupoid in Section 2 we were effectively taking the quotient of \mathbb{A}_K^* by the closure of $K^*(K_\infty^*)^\circ$. In terms of K -lattices this means that given a K -lattice (L, φ) we would have to identify not only all K -lattices $(kL, k\varphi)$ with $k \in (K_\infty^*)^\circ$, but also all K -lattices of the form $(kL, k\psi)$, where ψ is a limit point of the maps $u\varphi$ with $u \in \mathcal{O}^* \cap (K_\infty^*)^\circ$ in the topology of pointwise convergence.

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